Amplifying Sommerfeld precursors and producing a discontinuous index of refraction with gains and losses

Filipe J. Ribeiro and Marvin L. Cohen
Department of Physics, University of California, Berkeley, California 94720
and Materials Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720
(Received 9 April 2001; revised manuscript received 12 June 2001; published 19 September 2001)

We show that for specially designed linear dispersive media with one absorption line and one gain line the Sommerfeld precursors of a pulse can be amplified leading to an earlier detection of the signal. Also, we show that in some systems with one strong absorption line, a carefully placed gain resonance must induce a discontinuity in the imaginary part of the frequency dependent index of refraction and in the first derivative of its real part.

DOI: 10.1103/PhysRevE.64.046602 PACS number(s): 42.25.Bs, 42.50.Dv, 78.20.Ci

I. INTRODUCTION

The properties of the index of refraction $n(\omega)$ determine many features of the propagation of waves through dispersive media. The work of Brillouin [1] and Sommerfeld clarified some misunderstandings about the correct interpretation of phase, group, and energy velocity. Sommerfeld showed that, although group velocities can be larger than c, special relativity is not violated. In absorptive media, abnormal group velocities appear to be associated with high absorption. This led in the past to the commonly accepted but incorrect statement that group velocities greater than c or even negative have no physical meaning. In fact, several experiments have shown that light pulses can propagate faster than the speed of light in linear absorptive media without severe reshaping—apart from strong attenuation [2–6]—in agreement with a group velocity description. This does not violate special relativity theory because an analytical signal carries no information, as emphasized by Chiao [7]. An analytical signal is described by an analytical function for which all derivatives are well defined and continuous at any point in space. Therefore, by knowing all the derivatives at some point and by using a Taylor expansion, one can obtain information about the signal at *any* point in space. Information is carried by discontinuities in the signal or signal derivatives. These discontinuities always propagate with the speed of light and are consistent with causality. This means that, although it is possible to propagate a wave packet superluminally, it can never overtake the discontinuity or wave front.

There are other media where superluminality can occur with negligible absorption. In a medium with an *inverted* population of two-level atoms, instead of an absorption line we have a gain resonance line [8,9]. In such media the group velocity is less than c only for frequencies close to the gain resonance. In transparent regions of the frequency spectrum where the gain is negligible the phase and group velocities are larger than c. In fact, it is possible to prove from the Kramers-Kronig relations that in an inverted medium there are always transparent regions with superluminal phase and group velocities. [7] Another way of exploring gain media to get faster-than-c wave propagation is to use the nonlinearity associated with gain saturation. After amplifying the front of the wave, the saturated medium starts attenuating the back of the pulse, giving rise to a reshaping of the pulse that results in an effective faster-than-c propagation of the peak of the pulse [10,11]. This nonlinear effect is called *superluminous* wave propagation.

Recently Wang *et al.* [12] measured superluminal wave propagation for waves with carrier frequency between two very close gain lines. The measured group velocity was $v_g = -310c$. The time advance was about 62 ns for a 3.7 μs full width at half maximum (FWHM) Gaussian pulse. One could hope to achieve larger advance times compared with the FWHM. The difficulty resides in the fact that, although most of the spectrum of the pulse might be located in a transparent region, there are always nonzero small Fourier components at the gain resonance frequency originating from the front discontinuity. If there is no saturation, these components, however small, grow exponentially and become as strong as or stronger than the components at the carrier frequency. In fact, these high frequency Fourier components related to the discontinuity are responsible for what is known as the Sommerfeld precursor [1,13]. This is a high frequency and very low amplitude signal that precedes a light pulse—obeying relativity—when it goes through an absorptive medium. In this work we explore some consequences of these effects in media with one absorption line and one gain line. Also, we elaborate on an interesting discontinuity of the index of refraction.

II. SIGNAL AND MEDIUM

A typical realistic signal can be modeled by a pulse of frequency w_c with a Gaussian envelope centered at $t_c>0$ and with spread σ_s multiplied by a step function $\Theta(t)$ (no signal for $t<0$):

$$\Psi(t)=\cos(\omega_c t+\phi)e^{-(t-t_c)^2/\sigma_s^2}\Theta(t). \tag{1}$$

To represent a medium with one absorption line and one gain line we use the generalized Lorentz model for the dielectric function:

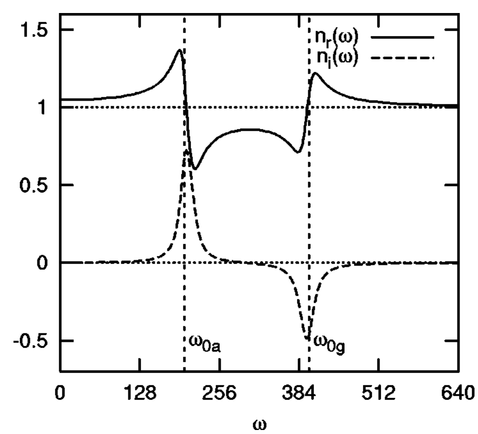

FIG. 1. Real part (solid line) and imaginary part (dashed line) of the refractive index $n(\omega)$ as a function of frequency, for a medium with one absorption line at $\omega_{0a}=200$ and one gain line at $\omega_{0g}=400$. The plasma frequency is $\omega_p=80$ and the widths of the resonances are $\gamma_a=20$ and $\gamma_g=25$ (all frequencies in arbitrary units). For the gain line $f=-1.2$.

$$\epsilon(\omega)=1+\frac{\omega_p^2}{\omega_{0a}^2-\omega^2-i\gamma_a\omega}+\frac{f\omega_p^2}{\omega_{0g}^2-\omega^2-i\gamma_g\omega}, \qquad (2)$$

where ω_p is the plasma frequency, ω_{0a} the absorption resonance frequency, γ_a the absorption linewidth, ω_{0g} the gain resonance frequency, γ_g the gain linewidth, and f a negative factor corresponding to the oscillator strength of a transition in an inverted medium.

To illustrate the features of this model in Fig. 1 we show a plot of the real and imaginary components of the refractive index $n(\omega)$ for $\omega_p=80$, $\omega_{0a}=200$, $\gamma_a=20$, $\omega_{0g}=400$, and $\gamma_g=25$ in arbitrary units of frequency, and for $f=-1.2$.

III. AMPLIFICATION OF SOMMERFELD PRECURSORS

A. Solving the wave equation

The time dependent amplitude of a signal propagating in the positive x direction at a specific point in space x_0 is given by

$$\Psi(x_0,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(x_0,\omega)e^{-i\omega t}d\omega. \qquad (3)$$

At some point $x>x_0$, we have

$$\Psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(x_0,\omega)\exp[ik(\omega)(x-x_0)-i\omega t]\,d\omega, \qquad (4)$$

where the wave vector is $k(\omega)=n(\omega)w/c$ and the index of refraction is $n(\omega)=\sqrt{\epsilon(\omega)}$ where $\epsilon(\omega)$ is the dielectric function and c is the velocity of light in vacuum. We assume that the magnetic permitivity $\mu(\omega)=1$.

To obtain the amplitude of the propagated signal all we need is $\Psi(x_0,t)$, its time Fourier transform, and the Green's function $\exp[ik(\omega)(x-x_0)]$, to then calculate the inverse Fourier transform of the resulting spectrum.

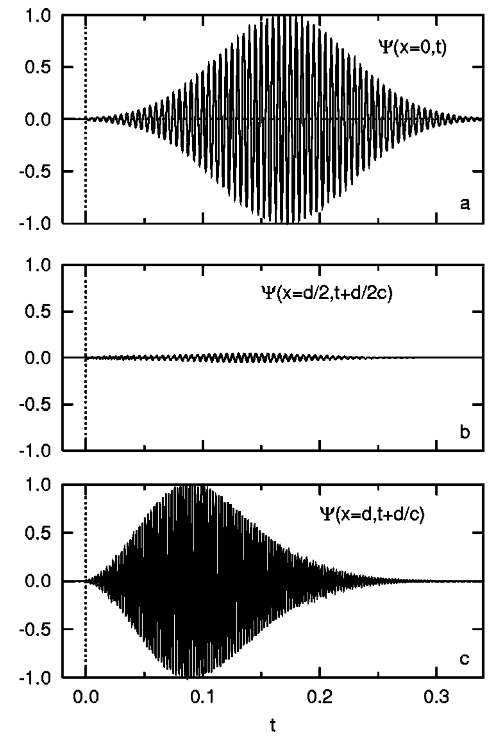

FIG. 2. Plots of the amplitude of a signal vs time (in arbitrary units) going through the medium described in the text. (a) Before entering the medium, (b) at the center of the medium, (c) immediately after exiting the medium. All the amplitudes have been plotted taking $t=0$ as the front edge of the signal for easier comparison. This front edge discontinuity always travels at the speed of light in vacuum, c.

For simplicity we assume that there is total transmission of the pulse.

B. Propagation

As mentioned above, a realistic signal is modulated by a step function. When propagating through an absorptive medium a signal will, in general, be reshaped, attenuated, and slown down by the medium. Because the signal has a front discontinuity, its Fourier spectrum will extend to infinity. There will always be low amplitude high frequency components in the pulse and because $n(\omega\rightarrow\infty)\rightarrow 1$ these components will be the least attenuated. So a very sensitive detector would first observe a very low amplitude signal of extremely high frequency. This is what is called the Sommerfeld precursor. After this initial signal another faint but low frequency signal will succeed it. This is called the Brillouin precursor [1,13] which we will not discuss further here. Only then will the original macroscopic pulse arrive, traveling with a group velocity less than c. If we now add a high frequency gain resonance the early high frequency Fourier components of the signal can be amplified. If the amplification is strong enough, the signal can be reshaped in such a way that a detector that is triggered by some threshold value of the amplitude will detect the pulse earlier. We then have achieved an effective advancement of the pulse.

In Fig. 2 we show plots of the time evolution of a signal

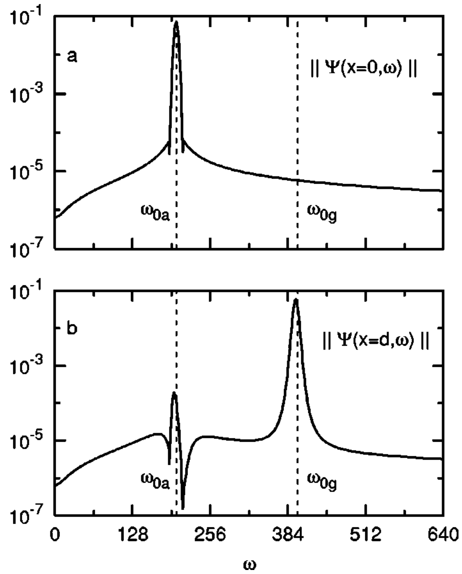

FIG. 3. Norm of the discrete Fourier transform of the signal (a) before and (b) after propagation through the medium.

with carrier frequency of 200 frequency units (the same as ω_{0a}, the absorption resonance) at three different values of x inside the dispersive medium. The total time of observation is 1 arbitrary unit of time consistent with the frequency unit. The total length of the medium in this case is $d=0.0076$ units of length ($c=1$ in this system of units). For easier comparison, the plots for $x=d/2$ [Fig. 2(b)] and $x=d$ [Fig. 2(c)] have been shifted in time so that the front edge of the signal, which always propagates at the speed of light, coincides for the three plots.

The plots show that at the middle of the medium ($x=0.0038$) the signal is highly damped due to the absorption line at the carrier frequency. At this point the amplification effects of the gain line are barely noticeable on a linear scale, and correspond to high frequency oscillations near the front edge. As the signal reaches the end of the dispersive medium, the components with frequencies close to the gain resonance ω_{0g} are amplified to a macroscopic level, in such a way that the final pulse has roughly the original amplitude.

In Fig. 3(a) we show, on a logarithmic scale, the norm of the Fourier transform of the signal before going through the medium. The original signal spectrum has a Gaussian peak around the carrier frequency corresponding to the real time Gaussian profile and, because of the front edge discontinuity, a slowly decaying tail that extends to infinity.

In Fig. 3(b) the norm of the Fourier transform of the signal is displayed after going through the medium. We see that the components of the signal for frequencies close to the carrier frequency have been attenuated by a factor of about 10^4, and that components with frequencies close to ω_{0g} have been amplified by roughly four orders of magnitude.

Note that the final shape of the pulse has a very weak dependence on the specific shape of the initial Gaussian pulse, since all of the major components are damped out by the absorption line. In fact, any step modulated pulse for which most of the Fourier components lie in the absorption line will produce the same output pulse. Also, it is important to note that the shape and frequency of the gain line determine the shape and frequency of the final pulse. The width of the gain line will be inversely proportional to the width of the output signal. If we have a narrow gain line we will get a broad output signal, but if we have a broad gain we will get a narrow output signal close to the front discontinuity. Actually, if the width of the gain line exceeds the width of the spectrum of the input signal, we get an effective advance of the pulse (see Fig. 2). In other words, the detection of the pulse will occur earlier than in the case of vacuum propagation.

IV. REFRACTIVE INDEX DISCONTINUITY

In this section we discuss an interesting property of media consisting of one strong absorption line and one strong gain line very close together.

From the wave equation we know that harmonic waves satisfy

$$k^2(\omega)=\epsilon\mu\frac{\omega^2}{c^2}=n^2(\omega)\frac{\omega^2}{c^2}. \tag{5}$$

In one space dimension there are two solutions, with $\mu=1$:

$$k(\omega)=\pm n(\omega)\frac{\omega}{c} \quad \text{where} \quad n(\omega)=\sqrt{\epsilon(\omega)}. \tag{6}$$

For the problem of transmission of a wave at an interface, the transmitted solution is usually given by $k(\omega)=+n(\omega)\omega/c$, where the real part of $k(\omega)$ is positive and represents a forward propagating wave. Close to an absorption line $k(\omega)$ will have a positive imaginary component, corresponding to an exponential decay inside the medium. If, on the other hand, the wave has a frequency that is close to a gain line, the imaginary part of $k(\omega)$ is negative and describes an exponential groth. In both cases Re $k(\omega)$ is positive. For low frequencies the imaginary part of $n(\omega)$ should be very small, and the real part should be positive. Also, for very high frequencies the index of refraction tends to the positive real value $+1$. For these reasons, when calculating $n(\omega)=\sqrt{\epsilon(\omega)}$, we must choose the root that has a positive real component. This root will correspond to the forward propagating wave. When the imaginary component of $\epsilon(\omega)$ is much smaller than the real component there is no ambiguity and the root is easy to find:

$$\epsilon=\epsilon_r+i\delta \quad \text{where} \quad |\delta|\ll\epsilon_r, \quad n\simeq\sqrt{\epsilon_r}+\frac{i\delta}{2\sqrt{\epsilon_r}}. \tag{7}$$

Problems arise when we have one strong absorption line and one strong gain line very close to each other. In Fig. 4(a) we plot the real (solid line) and imaginary (dashed line) components of the dielectric function for a system with the parameters $\omega_p=130$, $\omega_{0a}=300$, $\gamma_a=25$, $\omega_{0g}=350$, and $\gamma_g=20$ in arbitrary units of frequency and $f=-1$. For very low and very high frequencies $\epsilon(\omega)$ is of the form (7) and no problems arise. At intermediate frequencies changes occur. For these strong lines the real part of $\epsilon(\omega)$ reaches negative

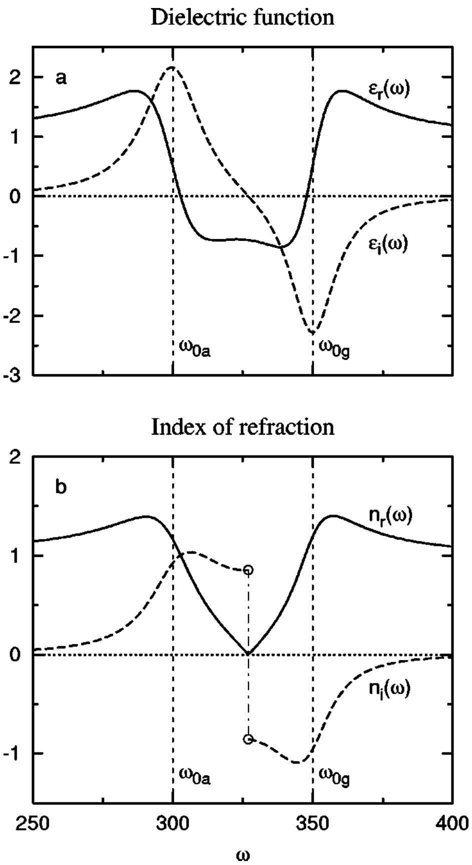

FIG. 4. Plot of the real (solid line) and imaginary part (dashed line) of the (a) dielectric function and (b) refractive index as a function of frequency (arbitrary units) for a medium with one strong absorption line and one strong gain line very close to each other. Parameters: $\omega_p=130$, $\omega_{0a}=300$, $\gamma_a=25$, $\omega_{0g}=350$, and $\gamma_g=20$ in arbitrary units of frequency and $f=-1$.

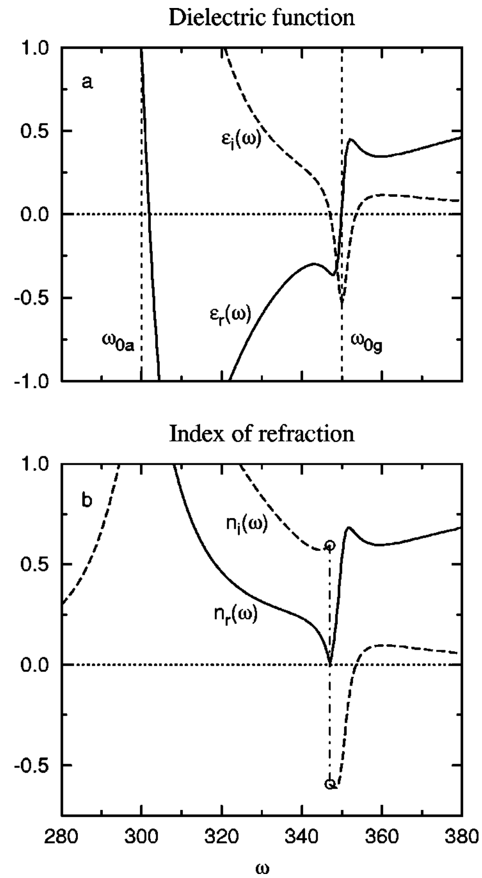

FIG. 5. Plot of the real (solid line) and imaginary part (dashed line) of the (a) dielectric function and (b) refractive index as a function of frequency (arbitrary units) for a medium with one strong absorption line and one weak gain line. Parameters: $\omega_p=180$, $\omega_{0a}=300$, $\gamma_a=20$, $\omega_{0g}=350$, and $\gamma_g=4$ in arbitrary units of frequency and $f=-0.177\,78$.

values for some frequencies. From Fig. 4(a) we see that there is a frequency ω_c between the two lines for which the imaginary part of $\epsilon(\omega)$ is zero and the real part is negative. At this frequency the refractive index must be strictly imaginary,

$$n(\omega_c)=\pm i\sqrt{|\epsilon(\omega_c)|}. \tag{8}$$

For frequencies $\omega<\omega_c$, $n(\omega)$ has a positive imaginary component and, for frequencies $\omega>\omega_c$, $n(\omega)$ has a negative imaginary component, while the real component is zero at ω_c and positive everywhere else. Effectively what we did was to choose a different branch of the square root so that we could keep $\mathrm{Re}\,k\geqslant 0$. If we imposed strict continuity we would have to change the positiveness of n either at very low frequencies or at very high frequencies which in this case would be unphysical. Therefore, a discontinuity in the imaginary part of the refractive index must exist. In Fig. 4(b) this discontinuity and the the discontinuity of the first derivative of $\mathrm{Re}\,n(\omega)$ are clear.

The consequences of this discontinuity are simple. For waves with frequencies below ω_c there is strong absorption, for waves with frequencies above ω_c there is strong amplification, and the transition from one regime to the other is abrupt. Experimentally, obtaining a strong gain line may be difficult. However, we find that one strong absorption line is enough to achieve this kind of discontinuity. Consider a strong absorption line centered at some frequency ω_{0a}. For frequencies above ω_{0a} the dielectric function has a negative real part and a positive imaginary part. At a frequency ω_x close to $\sqrt{\omega_{0a}^2+\omega_p^2}$, $\mathrm{Re}\,\epsilon(\omega_x)=0$ and $\mathrm{Im}\,\epsilon(\omega_x)>0$. If we now superimpose a relatively weak gain resonance centered close to ω_x, we can again get a discontinuity in the index of refraction. This is clearly seen in Fig. 5 where plots of (a) $\epsilon(\omega)$ and (b) $n(\omega)$ are shown for a system with parameters $\omega_p=180$, $\omega_{0a}=300$, $\gamma_a=20$, $\omega_{0g}=350$, and $\gamma_g=4$ in arbitrary units of frequency and $f=-0.177\,78$. In summary, to obtain a discontinuity in the index of refraction one requires that the imaginary part of $\epsilon(\omega)$ changes from positive to negative while the real part of $\epsilon(\omega)$ remains negative (see Figs. 4 and 5).

V. CONCLUSION

We demonstrate that a well designed linear medium with two resonances can be used to reshape a pulse so that the front is amplified and the back is attenuated. Compared to a pulse propagating in vacuum, the reshaped signal is detected earlier. Nevertheless, the front edge discontinuity will propagate at the velocity of light c both in vacuum and inside the medium. This can be interpreted as an effective amplification of the Sommerfeld precursors of a signal propagating through an absorptive medium. We emphasize that, although qualitatively similar to *superluminous* wave propagation, the effect described here is basically different because our system is fully linear.

The discontinuity of the imaginary part of the index of refraction is an interesting result. This is a consequence of the expected low and high frequency behavior of the index of refraction which determines the choice of the branch of the square root of the dielectric function. Experimentally one should see a very abrupt turn-on of propagation in the medium when the frequency rises above a certain critical frequency ω_c at which the dielectric function is negative and has zero imaginary part, making the index of refraction strictly imaginary.

ACKNOWLEDGMENTS

We would like to thank Professor Raymond Chiao for very interesting discussions and valuable comments. This work was supported by the NSF under Grant No. DMR0087088, and by the Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098. One of the authors (F.J.R.) was supported by FCT PRAXIS/BD/13465/97.

[1] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960).
[2] S. Chu and S. Wong, Phys. Rev. Lett. **48**, 738 (1982).
[3] S. Chu and S. Wong, Phys. Rev. Lett. **49**, 1293 (1982).
[4] B. Segard and B. Macke, Phys. Lett. **109A**, 213 (1985).
[5] A.M. Steinberg, P.G. Kwiat, and R.Y. Chiao, Phys. Rev. Lett. **71**, 708 (1993).
[6] A.M. Steinberg and R.Y. Chiao, Phys. Rev. A **51**, 3525 (1995).
[7] R. Y. Chiao, *Amazing Light, a Volume Dedicated to C. H. Townes on His Eightieth Birthday* (Springer, New York, 1996), pp. 91–108.
[8] R.Y. Chiao, Phys. Rev. A **48**, R34 (1993).
[9] E.L. Bolda, J.C. Garrison, and R.Y. Chiao, Phys. Rev. A **49**, 2938 (1994).
[10] E. Picholle, C. Montes, C. Leycuras, O. Legrand, and J. Botineau, Phys. Rev. Lett. **66**, 1454 (1991).
[11] D.L. Fisher and T. Tajima, Phys. Rev. Lett. **71**, 4338 (1993).
[12] L.J. Wang, A. Kuzmich, and A. Dogariu, Nature (London) **406**, 277 (2000).
[13] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, 1975).